Exploring high dimensional asset dependence through Vine Copulas

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Copulas

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In financial econometrics, you will encounter a vast array of financial models:

- Basic ARIMA models for the mean equation
- GARCH extensions to deal with heteroscedasticity
- Multivariate GARCH models that deal with dependence modeling

Theoretical problem arises when we talk about dependence:

- Capturing co-movement between financial asset returns with linear correlation has been the staple approach in modern finance since the birth of Harry Markowitz’s portfolio theory
- But linear correlation is only appropriate when the dependence structure (or joint distribution) follow a normal distribution

Copulas - flexible framework to model general multivariate dependence

Exploring high dimensional asset dependence through Vine Copulas
Goal

- To introduce to you an extension in the field of risk management
- Grasp basic concepts and generators within the field of copulas
  - Learn to walk, before we can run
  - Revisit your statistics
- Understand the field of copulas to such an extent that you might go on to do a PhD in this field ;-)
Fields where copulas are applied

- Quantitative finance
  - Stress-tests and robustness checks
  - “Downside/crisis/panic regimes” where extreme downside events are important
  - Pool of asset evaluation
  - Latest development: Vine Copulas
  - Hot research page [here](#)

- Civil engineering

- Warranty data analysis

- Medicine

*Exploring high dimensional asset dependence through Vine Copulas*
Introduction to copulas

- Copula stems from the latin verb copulare; bond or tie.
  - Regulated financial institutions are under pressure to build robust internal models to account for risk exposure
  - Fundamental ideology of these internal models rely on joint dependency among whole basket of mixed instruments
  - This issue can be addressed through the copula instrument
  - It functions as a linking mechanism between uniform marginals through the inverse cdf

- Copula theory was first developed by Sklar in 1959 Nelsen (2007).
**Introduction to copulas (Sklar)**

- Sklar’s theorem forms the basis for copula models as:
  - It does not require identical marginal distributions and allows for n-dimensional expansion

- Let $X$ be a random variable with marginal cumulative distribution function:
  - $F_X(x) = \mathcal{P}(X \leq x)$
  - Probability that random variable $X$ takes on a value less or equal to point of evaluation
  - $F_X(x) \sim U(0, 1)$
If we now denote the inverse CDF (Quantile function) as $F_X^{-1}$

- $U \sim U(0, 1)$ then $F_X^{-1}(U) \sim F(X)$

This allows a simple way for us to simulate observations from the $F_X$ provided the inverse cdf, $F_X^{-1}$ is easy to calculate

- Think, median is $F_X^{-1}(0.5)$

Let's have a look visually
Transformations

- PDF
- CDF
- $CDF^{-1}$
Transformations

- PDF
- CDF
- $CDF^{-1}$
Transformations

- PDF
- CDF
- $CDF^{-1}$
Definitions and basic properties

- Define the uniform distribution on an interval $(0,1)$ by $U(0,1)$, i.e.
the probability of a random variable $U$ satisfying $P(U \leq u) = u$ for $u \in (0,1)$

- This is the quantile function ($Q = F^{-1}$) Probability transformation
  implies that if $X$ has a distribution function $F$, then $F(X) \sim U(0,1)$ iff $F$
is continuous
**Definition (Copula):** A d-dimensional copula is the distribution function \( C \) of a random vector \( U \) whose components \( U_k \) are uniformly distributed

\[
C(u_1, \ldots, u_d) = P(U_1 \leq u_1, \ldots, U_d \leq u_d), (u_1, \ldots, u_d) \in (0, 1)^d
\]

Thus Sklar’s theorem states:

\[
C(F_1(x_1), \ldots, F_d(x_d)) = P(U_1 \leq F_1(x_1), \ldots, U_d \leq F_d(x_d))
\]

\[
= P(F_1^{-1}(U_1) \leq x_1, \ldots, F_d^{-1}(U_d) \leq x_d)
\]

\[
= F(x_1, \ldots, x_d)
\]
Joint distribution function:

- This represents the joint distribution function function \( F \) can be expressed in terms of a copula \( C \) and the marginal distributions \( (F_1, \ldots, F_d) \). Modeling them separately.

- **Easy Def:** A Copula is a function that couples the joint distribution function to its univariate marginal distribution.

- Dependence or correlation coefficient dependent on marginal distributions. This one to one mapping of correlation and dependence only works in case of elliptical joint distribution.

- For copulas, we use Kendall’s Tau - non-linear concordance measure.
Kendall’s Tau

- Let \((X_1, Y_1)\) and \((X_2, Y_2)\) be i.i.d random vectors, each with joint distribution function \(H\)

- Tau is then defined as the probability of concordance minus the probability of discordance

\[
\tau = \tau_{X,Y} = P((X_1 - X_2)(Y_1 - Y_2) > 0) - P((X_1 - X_2) - (Y_1 - Y_2) < 0)
\]

- Tau is the difference between the probability that the observed data are in the same order versus the probability that the observed data are not in the same order
A vine is a graphical tool for labeling constraints in high-dimensional probability distributions.

Regular Vines from part of what is known as pair copula construction.

Trees are constructed between copulas based on what is known as maximum spanning degree (or concordance measure).

Under suitable differentiability conditions, any multivariate density $F_{1...n}$ on $n$ variables may be represented in closed form as a product of univariate densities and (conditional) copula densities on any R-vine $V$. 

The R-vine copula density is uniquely identified according to Theorem 4.2 of @kurowicka2006:

\[
c(F_1(x_1), \cdots, F_d(x_d)) = \prod_{i=1}^{d-1} \prod_{e \in E_i} c_{j(e),k(e)|D(e)} \left( F(x_{j(e)}|x_{D(e)}) \right)
\]

(4)

- Introduction to [VineCopula](#)
- Website for the research [here](#)
Different structures

Vine copula specifications are based upon graph theory and more so Vines.

- This gives a lot of scope for the structure of the arrangement of assets.
- R-vine, D-Vine, C-Vine.
D-Vine, C-Vine and R-Vine

Each of the structures provide their own insight into the dynamics of the market

![Graph of Vine structures with nodes and edges labeled with parameters β₁₂, β₂₃, β₃₄, β₁₃₂, β₂₄₃, β₁₄₂₃.]]
D-Vine, C-Vine and R-Vine

Each of the structures provide their own insight into the dynamics of the market.

Tree 1

Copulas

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Applications

- C-vine offers a unique opportunity from centralized market player evaluation
  - CAPM
- Value at risk estimation of large portfolios bottom up
- Modeling complex dependence measures
A look into energy market dependence using Vine Copula

Exploring high dimensional asset dependence through Vine Copulas
Energy market through Vine Visualization

- Pre - GFC
- GFC
- Post - GFC

Copulas

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Energy market through Vine Visualization

- Pre - GFC
- GFC
- Post - GFC

Exploring high dimensional asset dependence through Vine Copulas
Quantifying dynamic dependence
Quantifying dynamic dependence

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# Final results

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**Table:** Mann-Whitey location test results
Conclusion

- Copulas act as a unique tool in to model non-conforming marginals that weren’t possible before.

- Vine Copulas have the ability to model complex relationships talks to their flexibility in their structuring.

- Informs on how assets are dependent - whether its tail dependence or general symmetric driving co-dependency.

- Opens the doors to practitioners (such as risk managers), to be better equipped in dealing with modern day finance.